

# Nonuniversality and Analytical Continuation in Moments of Directed Polymers on Hierarchical Lattices

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We prove the moments of the directed polymer partition function  $Z$ , using an exact position space renormalization group scheme on a hierarchical lattice. After sufficient iteration the characteristic function  $f(n) = \ln \langle Z^n \rangle$  of the probability  $\mathcal{P}(Z)$  converges to a stable limit  $f^*(n)$ . For small  $n$  the limiting behavior is independent of the initial distribution, while for large  $n$ ,  $f^*(n)$  is completely determined by it and is thus nonuniversal. There is a smooth crossover between the two regimes for small effective dimensions, and the nonlinear behavior of the small moments can be used to extract information on the universal scaling properties of the distribution. For large effective dimensions there is a sharp transition between the two regimes, and analytical continuation from integer moments to  $n \rightarrow 0$  is not possible. Replica arguments can account for most features of the observed results.

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**KEY WORDS:** Directed polymers; hierarchical lattices; disorder; moments.

## 1. INTRODUCTION

The directed polymer (DP) problem in a disordered environment<sup>(1)</sup> has by now become a paradigm of the statistical mechanics of random systems. Its connection to the Burgers' equation<sup>(2)</sup> describing the dynamics of surface growth in Eden clusters and ballistic deposition and to the seemingly unrelated problem of conduction in the strongly localized regime<sup>(3,4)</sup> makes this model extremely interesting. Briefly, in dimensions,  $d + 1 \leq 3$ , any amount of disorder is relevant, leading to nontrivial scaling with the length

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$L$  of the polymer. In particular, transverse fluctuations of the polymer are superdiffusive,  $\bar{x} \sim L^\zeta$  with  $\zeta > 1/2$ , while typical fluctuations in free energy scale as  $\Delta F \sim L^\omega$ . The exponents  $\zeta$  and  $\omega$  are related by the exponent identity  $\omega = 2\zeta - 1$ .<sup>(5)</sup> A renormalization group (RG) treatment<sup>(2)</sup> indicates that for  $d+1 > 3$ , a weak coupling phase at small disorder<sup>(6)</sup> is separated from a strong coupling phase at large disorder by a phase transition. The strong coupling exponents are known exactly only for  $d+1 = 2$ , where  $\zeta = 2/3$  and  $\omega = 1/3$ . There have been many recent studies of the exponents of the DP in the strong coupling phase.<sup>(7,8)</sup>

Perhaps the simplest analytic approach for getting the exact DP exponents in  $d+1 = 2$  is by a replica analysis of the moments of the partition function  $Z$ .<sup>(9)</sup> The replicated paths appearing in the average  $\langle Z^n \rangle$  can be regarded as the world-line of  $n$  attracting particles in one dimension, and is obtained from the ground-state energy of this system. As the  $n$ -particle bound-state energy scales as  $n(n^2 - 1)$ ,  $\langle Z^n \rangle = \langle Z \rangle^n \exp[-\rho n(n^2 - 1)L]$ . But  $\ln \langle Z^n \rangle = \sum_i (n^i/i!) C_i(\ln Z)$ , where  $C_i(\ln Z)$  is the  $i$ th cumulant of  $F = \ln Z$ , where the expansion is valid for *small*  $n$ . Assuming that the result calculated for integer  $n$  can be continued to  $n \rightarrow 0$ , the scaling of various cumulants is obtained from the replica result. The second cumulant must scale with a power less than one, while the third cumulant must scale as  $L$ . This is consistent with fluctuations scaling as  $\Delta F \sim L^{1/3}$ . Furthermore, it predicts that all higher-order cumulants of  $F$  must be absent. There have been a few recent studies of the full probability distribution  $\mathcal{P}(Z)$ . Starting from the replica result, Zhang<sup>(10)</sup> proposed an analytical form  $\mathcal{P}(F, L) \sim \exp(-a |F - \langle f \rangle L|^{3/2}/L^{1/2})$ . While this form captures the correct scaling of free energy fluctuations, it is symmetric about the average value, precluding the finite third cumulant required by the replica argument. This deficiency was remedied by Crisanti *et al.*<sup>(11)</sup> who generalize the above probability with different coefficients  $a_\pm$  depending on the sign of  $F - \langle f \rangle L$ . Recently Halpin-Healy<sup>(12,13)</sup> probed the moments and sampled the probability distribution for the free energy fluctuations and the lateral wandering of the DP. His transfer matrix investigation reveals that the probability distribution of  $F$  is indeed asymmetric, with second and third cumulants scaling as  $L^{2/3}$  and  $L$ , respectively.<sup>(9)</sup> Kim *et al.*<sup>(14)</sup> subsequently measured a nonzero *fourth cumulant* scaling as  $L^{4/3}$ , seemingly contradicting the replica result. The presence of a finite fourth moment suggests that one has to be careful about the interchange of the limits  $n \rightarrow 0$  and  $L \rightarrow \infty$  in the replica approach. The results of ground-state dominance are obtained by taking the  $L \rightarrow \infty$  limit first, while in the identification of cumulants the  $n \rightarrow 0$  limit must be performed first. The two limits, and all known results so far, can be brought into conformity by assuming  $\ln \langle Z^n \rangle = n \langle F \rangle L + g(nL^\omega)$ . For small arguments the scaling function has the form  $\lim_{x \rightarrow 0} = \sum_\alpha a_\alpha x^\alpha$ ,

thus generating all cumulants with appropriate scaling. For large arguments,  $g(x) \propto x^3$  in accordance with ground-state dominance. As we shall see later, there will be corrections to the large- $n$  limit due to the nonuniversality of high moments.

Hierarchical lattices provide a fertile testing ground for RG approaches, which can usually be performed *exactly* on such lattices.<sup>(15)</sup> Derrida and Griffiths<sup>(16)</sup> (DG) and Cook and Derrida<sup>(17)</sup> (CD) recently studied the DP problem on hierarchical lattices. They found a phase transition as a function of temperature in effective dimensions ( $2b = 2^{d_{\text{eff}}}$ ) greater than two; a high-temperature phase, where the quenched and annealed averages coincide, and a low-temperature phase, where  $\Delta F \sim L^\omega$  with  $\omega \approx 0.3$ , as expected for Euclidean lattices.<sup>(1)</sup> In the strong-disorder regime, Roux *et al.*<sup>(18)</sup> studied more general microscopic distributions on these lattices. They obtained two basins of attraction depending on the tail of the probability distribution function (PDF) for local disorder. If the tail of the distribution decays to zero sufficiently rapidly (as for Gaussian disorder), the fixed PDF is indeed asymmetric, and supports a third cumulant. They also *conjecture* that the exact exponent  $\omega = 1/3$  is reproduced for the hierarchical lattice with  $b = 2$ . However, this conjecture disagrees with the exact  $b = 1 + \varepsilon$  expansion of CD,<sup>(17)</sup> and is probably incorrect.<sup>(19)</sup> Nevertheless their results on the whole distribution may be qualitatively true for regular lattices, and other interesting properties of the distribution may be studied via hierarchical lattices.

The purpose of this paper is to derive information about the *moments* of the partition function by the exact RG on hierarchical lattices, and see whether the information on integer moments can be used to construct the universal scaling properties of  $\ln Z$ . We build upon the previous work of CD,<sup>(17)</sup> who studied the recursion relations for the low moments of  $\mathcal{P}(Z)$ . They found analytically that, for a Gaussian distribution of microscopic disorder, the  $n$ th moment undergoes a phase transition (for effective dimensions greater than two) at a temperature  $T(n)$ . In the high-temperature phase the quenched and annealed averages coincide and  $\langle Z^n \rangle \approx \langle Z \rangle^n$ . We generalize the recursion relations to arbitrary  $n$  and iterate them on a computer. The advantage of this scheme, compared to following the recursion relation for  $\mathcal{P}(Z)$  by Monte Carlo sampling of a histogram, is that it is tailored to provide direct information about the high moments and hence the tail of the distribution. (The Monte Carlo sampling would require an unreasonably large number of realizations.<sup>(12,13)</sup>) We follow the RG flow starting from the initial microscopic distribution of randomness, and obtain the final values of  $f^*(n) \equiv \ln \langle Z^n \rangle / L$ . We find two distinct regimes: For "small moments"  $f^*(n)$  converges to a fixed distribution proportional to  $n$ , while the "high moments" retain the form assigned by the initial

(microscopic) distribution. For  $b = 2$  there is a smooth crossover between the two regimes, and it appears that the information on high moments can be continued to infer the scaling of fluctuations in  $\ln Z$  (determined from the  $n \rightarrow 0$  limit). For  $b > 2$  the transition appears to be quite sharp, and the behavior of high moments merely reflects the nonuniversal character of the tail of the distribution for  $\mathcal{P}(Z)$ . The rest of the paper is organized as follows: In Section 2 the exact moment recursion relations are obtained on a diamond hierarchical lattice, and the limiting behaviors of the moments are discussed for various microscopic distributions. The nonuniversal behavior of the high moments and the tail of the distribution  $\mathcal{P}(Z)$  is emphasized. We also provide Monte Carlo results for the same lattice and compute the exponent  $\omega$ . In Section 3 the nonuniversality of the high moments and the appropriate interpretation of the replica argument are justified by examining the properties of the bound state. We end by noting the consequences of these results for other problems, such as the conduction in the strongly localized regime.

## 2. RECURSION RELATIONS

Consider the family of hierarchical lattices<sup>(15)</sup> constructed iteratively at each generation,  $m + 1$ , by replacing every bond of the lattice at the previous generation,  $m$ , by the motif of  $2b$  bonds shown in Fig. 1a. The ratio of the number of bonds at successive generations can be used to define an effective dimension  $d_{\text{eff}}$  via  $2b = 2^{d_{\text{eff}}}$ . We report results for lattices with  $b = 2$  and  $b = 3$ . The problem of directed polymers on these lattices reduces to determining the stable laws for combining random variables in a nonlinear fashion described below. The DP partition function is given by

$$Z = \sum_{\Gamma} \exp(-E_{\Gamma}/T) \quad (2.1)$$

where  $E_{\Gamma}$  is the energy of the path  $\Gamma$ , the summation is over all possible directed paths between the endpoints (see Fig. 1b), and  $T$  is the temperature. The energy  $E_{\Gamma}$  of each walk is obtained from

$$E_{\Gamma} = \sum_{i_{\Gamma}} \varepsilon_{i_{\Gamma}} \quad (2.2)$$

where  $\varepsilon_{i_{\Gamma}}$  are random numbers assigned to the bonds of the lattice from a predetermined probability distribution. The objective is to determine the asymptotic distribution function for  $Z$  as  $L = 2^{m-1} \rightarrow \infty$ , where  $L$  is the length of directed paths at generation  $m$ . One can construct a recursion relation for the partition function in the following way: Let us take the case

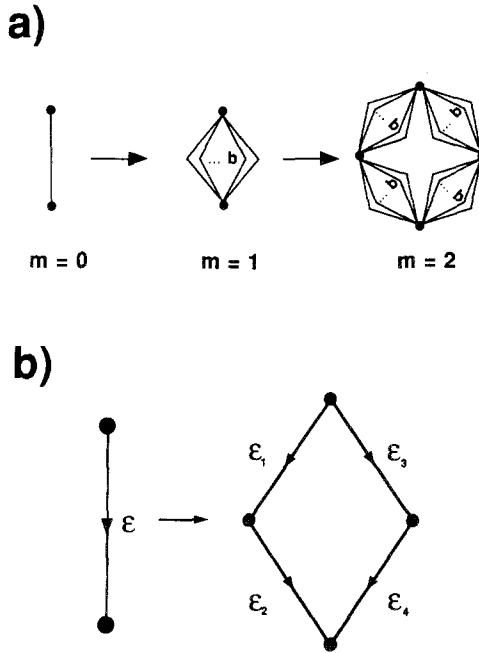


Fig. 1. (a) Recursive construction of a hierarchical lattice: Each bond at a given generation is converted to a motif of  $2b$  bonds to generate the next generation. (b) The partition function  $Z$  of the directed polymer is obtained from the sum over all possible directed walks as illustrated.

of  $b = 2$ , where the zeroth generation is a single bond whose energy is  $\epsilon$ . There is only one walk, and the partition function is (see Fig. 1b)

$$Z_0 = \exp(-\epsilon/T) \tag{2.3}$$

The first generation has four bonds ( $2b$  in general), each with an independent random variable  $\epsilon_i$ . As there are two possible paths, the resulting partition function is

$$Z_1 = Z_0^{(1)} Z_0^{(2)} + Z_0^{(3)} Z_0^{(4)} \tag{2.4}$$

where the superscripts label the bonds, the subscripts the generation number. This recursive relation in the general case (arbitrary  $b$ ) is

$$Z_{m+1} = Z_m^{(1)} Z_m^{(2)} + Z_m^{(3)} Z_m^{(4)} + \dots + Z_m^{(2b-1)} Z_m^{(2b)} \tag{2.5}$$

The quenched disorder enters at the lowest (microscopic) generation,  $m = 1$ . We shall focus on the evolution of the integer moments,

$$\int P(\epsilon) Z_m^n d\epsilon = u_m(n) \tag{2.6}$$

where the argument of  $u$  stands for the moment index, and the subscript indicates the generation number. The normalization of the probability distribution requires

$$u_m(0) = 1 \quad (2.7)$$

The recursion relation for the  $n$ th moment is obtained as the sum over all possible ways to propagate  $n-s$  paths on one branch and  $s$  paths on the other ( $b=2$ ), i.e.,

$$u_{m+1}(n) = \sum_{s=0}^n \frac{n!}{(n-s)! s!} [u_m(s)]^2 [u_m(n-s)]^2 \quad (2.8)$$

It is easy to generalize the recursion relations in Eq. (2.8) to arbitrary  $b$ . The results for a uniform lattice (trivial path counting) are readily recovered starting from the condition  $u_1(n) = 1$  for all  $n$ . For the case of an arbitrary initial distribution we set

$$\int P(\varepsilon) Z_1^n d\varepsilon = u_1(n) = \exp[f_1(n)] \quad (2.9)$$

where  $f_0(n) = \alpha n + \beta n^2 + \gamma n^3 + \dots$  is the characteristic function of the microscopic distribution written as an arbitrary polynomial in  $n$ . We expect that under iteration by the above recursion relations the initial distribution converges to a fixed function  $f^*(n)$  with the moments given by

$$u_m(n) = 2^{(L-1)n} \exp[Lf^*(n)] \quad (2.10)$$

[The first factor on the right-hand side of Eq. (2.10) is just the number of paths at the  $m$ th generation.]

The numerical results are qualitatively different for the cases  $b > 2$  and  $b \leq 2$ . The result for  $b=3$  and different initial distributions of randomness are given in Fig. 2. [The vertical axis is scaled by  $L=2^{m-1}$  in order to isolate  $f^*(n)$ .] Starting with  $f(n) = \beta n^k$ , we first see a linear behavior in  $n$  which then crosses over to a dependence identical to the one used for the microscopic distribution. The case  $k=2$  corresponds to a Gaussian distribution, where  $\beta$  is related to the variance of the microscopic probability distribution (disorder strength  $\sigma^2$ ) by  $\beta = \sigma^2/2T^2$ . Figure 3 indicates that the point of crossover between the two regimes depends on  $\beta$ . In fact the transition between the two behaviors is very sharp. The various moments have different critical temperatures: on decreasing  $T$ , the higher moments slip into the nonuniversal low-temperature behavior before the lower moments (which stay in the linear regime). To the precision of one part in

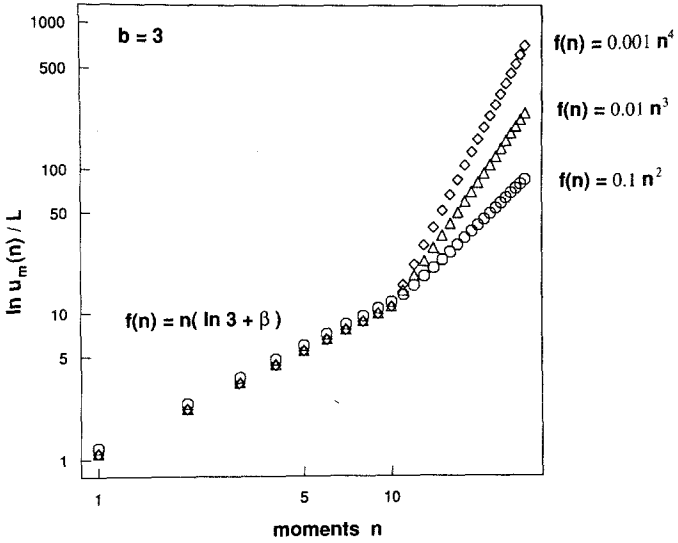


Fig. 2. Fixed values of log moments for  $b=3$ , starting with initial distributions,  $f_1(n) = 0.1n^2$  (circles),  $0.01n^3$  (triangles), and  $0.001n^4$  (diamonds).

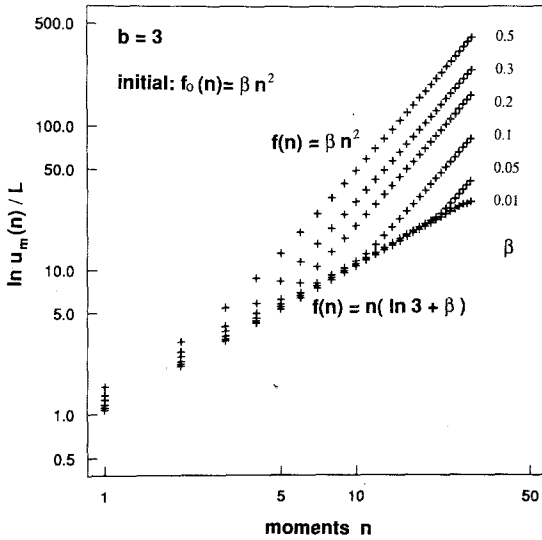


Fig. 3. Same as Fig. 2, but starting with  $f_1(n) = \beta n^2$ . Note the sharp change between the linear and the quadratic behaviors.

$10^{12}$  achieved in our simulations, the results for *all* initial distributions can be summarized by the remarkably simple formula

$$\lim_{m \rightarrow \infty} \frac{\ln u_m(n)}{L} = \begin{cases} (\ln b + \beta)n & \text{for } (\ln 2 + \beta)/\beta < n \\ f_1(n) & \text{otherwise} \end{cases} \quad (2.11)$$

The results for  $b = 2$  and  $k = 2$ , depicted in Fig. 4, are qualitatively different in that the sharp transition between the two regimes is replaced by a smooth crossover. To understand this crossover, we subtracted the linear term of  $(\ln b + \beta)n$ , corresponding to the annealed free energy. The results, depicted in Fig. 5a, show a leftover nonlinear dependence on  $n$ . The figure focuses on the dependence as  $n$  decreases for three values of temperature. Figure 5b shows the limiting slopes (at small  $n$ ) of the data in Fig. 5a and also includes some lower temperatures. It is interesting to note an effective power law dependence of  $n^\delta$  with  $\delta = 2.83 \pm 0.02$ , close to the  $n^3$  dependence expected for the continuum limit. This is a lower power than expected from the results of DG,<sup>(16)</sup> where  $1/\delta \approx 0.3$ . As the recursion relations are only valid for integer  $n$ , one cannot be absolutely certain that the power laws continue to the important  $n \rightarrow 0$  limit. However, the simplest scenario (see Section 3) is that they indeed do so, but that various corrections to scaling make the effective exponents computed at integer  $n$  less reliable. This difficulty was also encountered in the variational approach of Blum *et al.*<sup>(20)</sup> to the problem of a complex DP.<sup>(4,10)</sup>

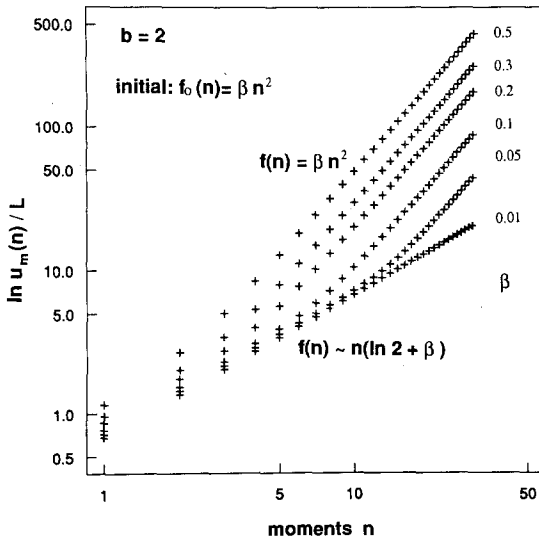


Fig. 4. Same as Fig. 3, but for  $b = 2$ .



One additional interesting feature of the numerical results summarized by Eq. (2.11) is the prediction that the position of the transition for the moments is given by the condition

$$\frac{\ln 2 + \beta}{\beta} = n \quad (2.12)$$

It follows that for Gaussian disorder with  $\beta = 1/2T^2$  and  $n = 2$ , one has  $T_c = (2 \ln 2)^{-1/2}$ , the *lower bound* proposed by Cook and Derrida.<sup>(17)</sup> This lower bound corresponds to the temperature at which the entropy computed from the annealed free energy turns negative. This coincidence is certainly worth pursuing, and has been the subject of a recent study by Evans and Derrida.<sup>(21)</sup>

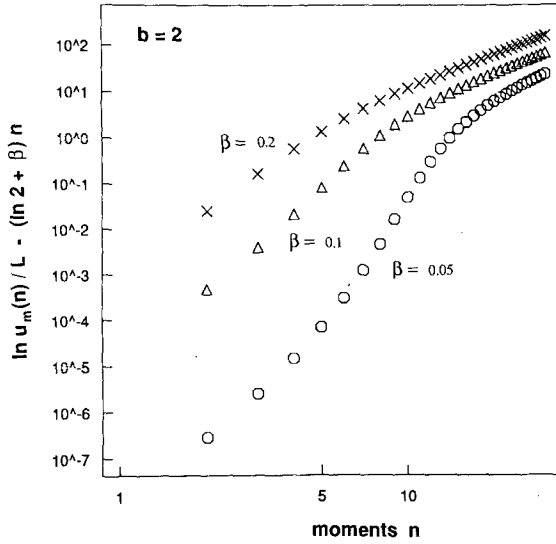
Monte Carlo studies were also carried out on a  $b = 2$  hierarchical lattice for the case of a Gaussian microscopic distribution of the energies. We carried out more than 100,000 samplings of lattices of generation  $m = 9$ . While the behavior of the low moments was readily verified, the higher moments were difficult to access due to the difficulties of sampling the tail of the distribution. In this respect the exact RG procedure has the important advantage of preserving the tails exactly. The Monte Carlo data confirm the expected linear dependence for the free energy  $\langle \ln Z \rangle = (0.13 \pm 0.01)L$ , and the scaling of fluctuations  $\langle (\ln Z)^2 \rangle - \langle \ln Z \rangle^2$  as  $L^{2\omega}$ . The measured  $\omega = 0.29 \pm 0.01$  is lower than the known result for Euclidean lattices, and disagrees with the conjecture of Roux *et al.*<sup>(18)</sup> that  $\omega = 1/3$  for this lattice. This also points to a higher value  $\delta$  calculated above from the limited knowledge of integer moments. The sampled distribution also served to check the linear and nonuniversal regimes of  $\ln \langle Z^n \rangle$  derived by the exact renormalization scheme (only in the small portion allowed by sampling).

### 3. REPLICA ANALYSIS

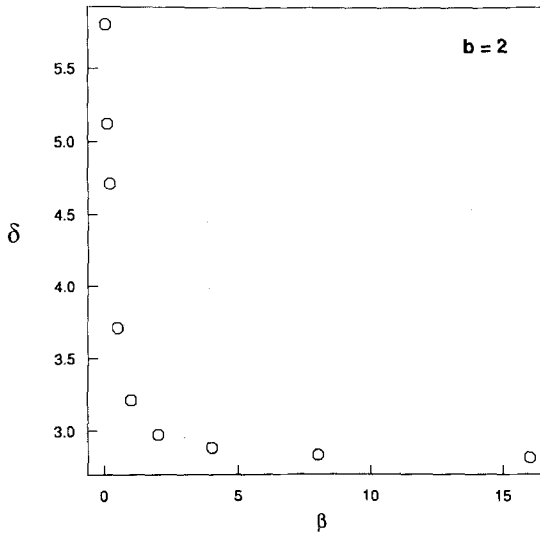
To gain some understanding of these results, we make a comparison to the replica calculations of moments in a  $(d+1)$ -dimensional Euclidean space. The replicated walks in  $\langle Z^n \rangle$  can be regarded as world-lines of  $n$  bosons in  $d$  dimensions. In a *continuum approximation* the bosons are subject to a Hamiltonian<sup>(9)</sup>

$$-\mathcal{H}_n = \frac{\sigma^2}{2T^2} + \gamma \sum_{\alpha}^n \frac{\partial^2}{\partial \mathbf{x}_{\alpha}^2} + \sigma^2 \sum_{\beta < \alpha} \delta^d(\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}) \quad (3.1)$$

The  $n$  dependence of the ground-state energy can be estimated by the following argument.<sup>(10)</sup> Consider a variational wave function describing a



(a)



(b)

Fig. 5. (a) Behavior of low moments for  $b=2$ , after subtracting  $(\ln 2 + \beta)n$ . The remainder shows a nonlinear behavior for small  $n$ . Nontrivial higher powers of  $n$  are observed in the limit  $n \rightarrow 0$ . (It is not possible to include  $n=1$  because its value is exactly  $\ln 2 + \beta$  to machine precision). (b) The exponent for the nonlinear behavior is observed in the limit  $n \rightarrow 0$ . The value  $\delta = 2.83$  is close to the continuum result of  $\delta = 3$ .

bound state confined to a size  $R$ . Estimating the kinetic and potential energy terms in Eq. (3.1) gives

$$\varepsilon_n \approx \frac{d\gamma n}{R^2} - \frac{\sigma^2 n(n-1)}{R^d} \quad (3.2)$$

Minimizing this expression with respect to  $R$  gives different behavior, depending on  $d$ . For  $d < 2$  there is a minimum at  $R^{2-d} \propto l^*/(n-1)$ , with  $\varepsilon_n \propto \gamma n [(n-1)/l^*]^{2/(2-d)}$  and  $l^* \propto \gamma/\sigma^2$ . As  $\sigma^2 \rightarrow 0$ , the size  $R$  diverges, and thus the use of the continuum approximation is internally self-consistent. However, as  $n$  increases, the particles become more tightly bound, and eventually the continuum approximation breaks down when  $R$  approaches the lattice spacing. In this limit the  $n$  replicas follow the same trajectory and hence feel the corresponding moment of the local distribution chosen for the bonds [i.e.,  $-\varepsilon_n = f_1(n)$ ]. This clearly leads to a nonuniversality of the high moments that does not go away as the size  $L$  is increased.<sup>(22)</sup> This nonuniversality is also present on the hierarchical lattice, and accounts for the crossover point  $n_c$  in Fig. 4. For  $d=2$  the crossover occurs for  $n_c \propto 1/\sigma^2$ . This is approximately the behavior observed for  $b=2$  on the hierarchical lattice, as shown in Fig. 6. Since  $n_c$  can be made arbitrarily large, we can get valuable information about the universal ( $n \rightarrow 0$ ) part of the distribution by studying *finite integer* moments as in Fig. 5.

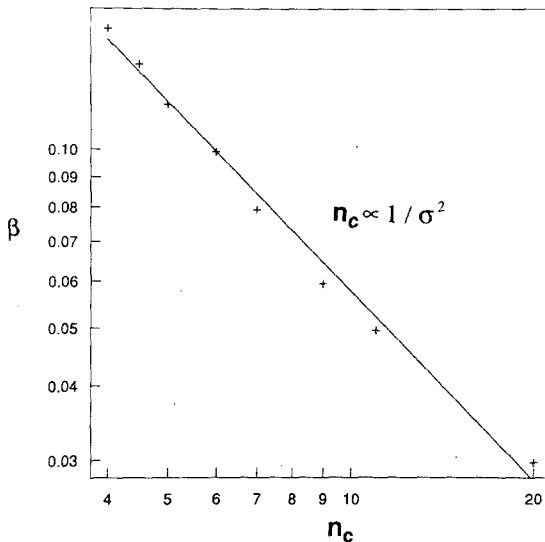


Fig. 6. Scaling of the crossover point  $n_c$  between linear and quadratic behaviors from data similar to Fig. 4. The fit is to the straight line predicted for a  $(d+1)$ -dimensional Euclidean lattice.

For  $d > 2$ , Eq. (3.2) has two minima at  $R \rightarrow \infty$  and  $R \rightarrow 0$ . The former corresponds to unbound particles and the latter to a collapse of all particles to a single point. The collapse is of course modified by lattice effects, and the continuum approximation is no longer valid. In fact, if all paths follow the same trajectory,  $f^*(n) = f_1(n)$  in Eq. (2.10). The minimum at  $R = 0$  now has a finite energy which can be compared to the zero energy of the unbound state. The crossing of these two levels leads to a *discontinuous* phase transition, controlled by the nonuniversal details of the microscopic randomness described by  $f_1(n)$ . The unbound phase has an energy that increases linearly with  $n$  as in Eq. (2.11). This is the "free" phase of the DP in which the annealed and quenched averages are identical. It is interesting that the numerical results in Eq. (2.11) also suggest that in the bound state the paths are fully collapsed. The behavior of the moments is nonuniversal in this state, merely reflecting properties of the microscopic randomness distribution. There is also a phase transition in the probability distribution for  $\ln Z$  in  $b > 2$ . For sufficiently large disorder the distribution becomes broad and exhibits *universal* scaling properties, which have been numerically investigated.<sup>(12)</sup> Clearly the knowledge of the moments in Eq. (2.11) does not provide any insight into the scaling of  $\ln Z$  in this case. Another interesting consequence is that the bulk and tail of the distribution can renormalize independently. For small  $\beta$ , while the bulk evolves toward a delta function, the tails will be governed by the microscopic distribution.

There have been several replica treatments of real<sup>(23)</sup> and complex variants<sup>(10,8)</sup> of the DP problem. The above results indicate that the analytical continuation from integer  $n$  to  $n \rightarrow 0$  must be taken with great care. It is justified only if this limit can be taken without encountering any singularities. This difficulty could account for the disagreements between the results of refs. 10 and 8 and various numerical work.<sup>(24)</sup> An illuminating application of this method is given in recent variational and  $1/d$  treatments of the problem in ref. 8. Different analytical forms are identified for  $n > 1$  and  $n < 1$ . The forms for  $n < 1$  exhibit a universal nonlinear behavior that provides the scaling of  $\ln Z$ .

An interesting variation of the DP occurs for strongly localized electrons. The overlap integral between two impurities is obtained by a Feynman sum over all directed paths connecting the impurities.<sup>(4,22)</sup> This overlap integral is essentially the conductance associated with the hop. The only difference between this model and the regular DP is that bond randomness can take both positive and negative values. Transfer matrix studies<sup>(22)</sup> indicate that this model is in the same universality class as the regular DP (for a different point of view see ref. 10). One can thus expect the behavior of the probability distribution for the conductance of an

insulator to show some of the features discussed above for the partition sum of the DP. Altshuler *et al.*<sup>(25)</sup> (AKL) have computed the moments of the conductance using an extended nonlinear  $\sigma$  model. In the metallic regime they find a probability distribution that is close to a Gaussian except for long log-normal tails. At the metal-insulator transition in  $2 + \epsilon$  dimensions sufficiently large moments diverge in  $L$ , indicating nonuniversal behavior in the tail of the distribution, reminiscent of our results. Shapiro<sup>(26)</sup> has pointed out that the divergence of these moments is not inconsistent with universality, and proposes a distribution that reproduces the moments of AKL. The key point is that, for distributions with power-law tails, large enough moments are divergent. (Conversely, if the  $n$ th moment increases faster than  $n!$ , the probability distribution cannot be uniquely inferred.<sup>(27)</sup>) Shapiro suggests that the nonuniversality of the moments found by AKL only reflects that these are “bad” scaling variables, and cannot be used to confirm or reject scaling. Another possibly related result is by Schreiber and Grussbach,<sup>(28)</sup> who find multifractal behavior at the metal-insulator transition in three dimensions. As a result the moments of conductance are rather similar to those we found above. Further work is in progress to make the connections between these systems more transparent.

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